

# On the construction of pseudo-Hermitian Hamiltonians by means of similarity transformations

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April 21, 2016

## Abstract

We generalize a recently proposed approach for the construction of pseudo-Hermitian Hamiltonians with real spectra. Present technique is based on a simple and straightforward similarity transformation of the coordinate and momentum.

## 1 Introduction

In a recent paper Miao and Xu [1] derived two pseudo-Hermitian Hamiltonians that have real spectra because they are isospectral to Hermitian Hamiltonians. The approach is based on the Heisenberg equations of motion for the coordinate and momentum. The similarity transformation between the Hermitian and pseudo-Hermitian Hamiltonians is expressed in terms of an infinite series of either the coordinate (first model) or momentum (second model).

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The purpose of this short article is to show that one can easily derive those pseudo-Hermitian Hamiltonians by means of a considerably simpler and more straightforward technique developed recently [2].

## 2 The similarity transformation

Let  $x$  and  $p$  be the dimensionless coordinate and momentum, respectively, that satisfy the commutation relation  $[x, p] = i$ . If  $f(x)$  is a differentiable real function of  $x$  we have

$$e^f p^2 e^{-f} = (p + i f')^2 = p^2 - f'^2 + i (f' p + p f'), \quad (1)$$

where the prime denotes differentiation with respect to  $x$ . Therefore

$$e^f (p^2 + V + f'^2) e^{-f} = p^2 + V + i (f' p + p f'), \quad (2)$$

where  $V = V(x)$ , is an arbitrary function of  $x$ . If we choose  $V(x)$  and  $f(x)$  so that  $H_H = p^2 + V(x) + f'^2$  supports real eigenvalues then the non-Hermitian Hamiltonian  $H = p^2 + V(x) + i [f'(x)p + p f'(x)]$  will exhibit exactly the same real eigenvalues [2].

If we now take into account that

$$e^{-f} p^2 e^f = (p - i f')^2 = p^2 - f'^2 - i (f' p + p f'), \quad (3)$$

then it follows from equation (2) that

$$e^{-f} (p^2 + V + f'^2) e^f = p^2 + V - i (f' p + p f') = e^{-2f} [p^2 + V + i (f' p + p f')] e^{2f}. \quad (4)$$

In other words,  $H$  is  $\eta$ -pseudo-Hermitian [3–5]

$$H^\dagger = \eta^{-1} H \eta, \quad (5)$$

where  $\eta = e^{2f}$ .

If  $\psi$  is an eigenfunction of  $H_H$  with eigenvalue  $E$ , then  $e^f \psi$  is eigenfunction of  $H$  with the same eigenvalue and we should choose  $V(x)$  in such a way that  $e^f \psi$  is square integrable in the case of a bound state.

The technique proposed by Miao and Xu [1] is a particular case of the one developed above when

$$f(x) = - \sum_{k=0}^{\infty} \frac{c_k}{k+n+1} x^{k+n+1}, \quad n \geq 0. \quad (6)$$

Present approach is somewhat more general because the function  $f(x)$  does not necessarily have to be of this rather restricted form. For example, if we choose

$$V(x) = \frac{3D}{4} (1 - e^{-\alpha x})^2, \quad D > 0, \quad \alpha > 0, \quad (7)$$

and

$$f(x) = \frac{\sqrt{D}}{2} (x + \alpha^{-1} e^{-\alpha x}), \quad (8)$$

then we obtain a pseudo-Hermitian operator with the spectrum of the Morse oscillator [6]

$$H_H = p^2 + D (1 - e^{-\alpha x})^2. \quad (9)$$

Note that this model exhibits discrete spectrum for  $0 < E < D$  and continuous one for  $E > D$ .

In order to derive the second model we simply reverse the roles of the coordinate and momentum and write the similarity transformation in terms of a function  $g(p)$ :

$$e^g x^2 e^{-g} = (x - ig')^2 \quad (10)$$

In this case we should choose a suitable function of the momentum  $V(p)$  in order to obtain an Hermitian Hamiltonian with real spectrum [1].

It is quite easy to generalize the results above to a quantum-mechanical system described by a set of  $N$  coordinates  $\mathbf{x} = (x_1, x_2, \dots, x_N)$  and their conjugate momenta  $\mathbf{p} = (p_1, p_2, \dots, p_N)$ . In this case we obtain the isospectral Hamiltonians

$$\begin{aligned} H_H &= \mathbf{p} \cdot \mathbf{p} + V(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \nabla f(\mathbf{x}) \\ H &= \mathbf{p} \cdot \mathbf{p} + V(\mathbf{x}) + i [\nabla f(\mathbf{x}) \cdot \mathbf{p} + \mathbf{p} \cdot \nabla f(\mathbf{x})]. \end{aligned} \quad (11)$$

### 3 Conclusion

In this short article we have shown that an approach for the construction of pseudo-Hermitian Hamiltonians with real spectra developed recently [2] is simpler and more straightforward than the one proposed by Miao and Xu [1] in two main aspects. First, our technique does not require resorting to the Heisenberg equations of motion; second, it avoids the use of a particular power series for the function that appears in the similarity transformation. In this way, present method facilitates the construction of pseudo-Hermitian Hamiltonians that are isospectral to Hermitian ones.

### References

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